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1998 J. Phys. A: Math. Gen. 31 2491

(http://iopscience.iop.org/0305-4470/31/10/021)

J. Phys. A: Math. Gen. **30** (1997) 2491–2501. Printed in the UK PII: S0305-4470(97)82953-8

An inverse scattering transform for the Landau–Lifshitz equation for a spin chain with an easy plane

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Received 1 April 1997, in final form 6 August 1997

Abstract. The Landau–Lifshitz equation for a spin chain with an easy plane in the case of spin non-flip is solved by the method of inverse scattering transform. To avoid complexity caused by the Riemann surface of the usual spectral parameter, a particular parameter k is introduced. After performing a gauge transformation corresponding to $|k| \to \infty$, the resulting Lax pair is independent of particular solutions in this limit. An inverse scattering transform is then developed in terms of k. A system of linear equations is derived in the reflectionless case. An expression of the gauge transformation and hence expressions of multi-soliton solutions are found explicitly by using the Binet–Cauchy formula. As an example, an explicit expression of the 1-soliton is given in terms of elementary functions of x and t.

1. Introduction

In the last 20 years, the Landau–Lifshitz (L–L) equation for a spin chain with an easy plane has attracted much attention [1]. There exist localized permanent shape solutions for it, as mentioned in [2] and evident also from [3] in the appropriate limit. However, the equation is hard to solve exactly [4]. It is impossible to find, as mentioned in [5], a general stationary (i.e. depending on $x - vt$) solution. Solutions of this type given in previous works [6, 7] do not satisfy the equation even in the approximation of first-order anisotropy.

Hirota [8] has shown the possibility of the existence of many soliton solutions for the general case with two anisotropies, but explicit soliton solutions for the L–L equation with an easy plane were not constructed [9]. Another attempt [10] was made to reduce the equation to an approximate equation, but the solution which was found could not be considered as an approximate solution of the L–L equation with an easy plane, since it does not satisfy the equation in the approximation of first-order anisotropy.

It was attempted to solve the equation by the inverse scattering transform [11, 12]. However, in addition to the complexity due to the Riemann surface, required by the double-valued function of the usual spectral parameter, the equation has no common property in that the Lax pairs are independent of the particular solutions of the equations in some limiting values of the spectral parameters, just as for most nonlinear equations that have been solved by the inverse scattering transform. This means that one needs an additional idea to face the induced difficulty, but at present suggestions are not forthcoming. Borovik *et al* [11, 12] attempted to transform the equation to a new one which had the above-mentioned property by means of an appropriate qauge transformation, such as the L–L equation of an isotropic spin chain, but was unsuccessful.

Recently, the exact soliton solutions to the equation with spin non-flip found by means of the method of Darboux transformation matrix, were reported [13–15]. However, with this

0305-4470/97/102491+11$19.50

method, it is impossible to include the contribution due to the continuous spectrum of the spectral parameter. If one considers perturbation theory for the equation with corrections, such as that with an applied field which is more important in practice, a general theory with terms of the continuous spectrum as a starting point is necessary.

In this paper, an inverse scattering transform is developed for the L–L equation with an easy plane in the case of spin non-flip. To avoid the double-valued function of the usual spectral parameter, one may introduce an affine parameter, as in the nonlinear Schrödinger (NLS) equation of normal dispersion with non-vanishing boundary values [16]. However, it is unsuitable in the present case because of the lack of the above-mentioned common property of the solved equations. To develop an inverse scattering transform one needs to overcome this difficulty. It is necessary that a gauge transformation be introduced [17] and chosen such that the resulting equation does have the above-mentioned common property. By introducing a particular parameter k (see later equation (5)), first that the complexity due to the Riemann surface in terms of the usual spectral parameter is avoided, and second the gauge transformation is determined by the Lax equations corresponding to $|k| \to \infty$. Then for the resulting equation analyticities of the Jost solutions as functions of k are derived. An equation of the inverse scattering transform is then deduced. In the case of no reflection, after determining the expression of the gauge transformation, expressions for multi-soliton solutions of the equation with an easy plane are obtained explicitly by solving the linear algebraic equations with the aid of the well known Binet–Cauchy formula [18]. This method is more effectual than that of the Darboux transformation matrix [13, 14] and the asymptotic behaviour of the N-soliton solution in the limit $|t| \to \infty$ can be obtained simply. The 1-soliton solution is found explicitly in terms of elementary functions of x and t. These are the same functions as those in the method of the Darboux transformation matrix [13]. The present inverse scattering transform method includes the contributions due to the continuous spectrum of the spectral parameter. It, therefore, provides a suitable basis for developing a perturbation theory for the equation with corrections.

2. The L–L equation with an easy plane

The L–L equation for a spin chain with an easy plane is

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{xx} + \boldsymbol{S} \times \boldsymbol{J}\boldsymbol{S} \qquad |\boldsymbol{S}| = 1 \tag{1}$$

where the diagonal matrix $\boldsymbol{J} = \mathrm{diag}(0, -16\rho^2, 0)$, which characterizes the easy plane, the 31-plane. Here ρ is a positive constant and 16 is introduced for later convenience. In the case of spin non-flip we assume that

$$\boldsymbol{S} \to \boldsymbol{S}_0 = (0, 0, 1) \qquad \text{as } |x| \to \infty. \tag{2}$$

The Lax pair of equation (1) is given by [3]

$$L = -\mathrm{i}\mu S_2\sigma_2 - \mathrm{i}\lambda(S_3\sigma_3 + S_1\sigma_1) \tag{3}$$

$$\begin{aligned} M = {} & \mathrm{i}2\lambda^2 S_2\sigma_2 + \mathrm{i}2\lambda\mu(S_3\sigma_3 + S_1\sigma_1) - \mathrm{i}\lambda(S_1S_{2x} - S_2S_{1x})\sigma_3 \\ & -\mathrm{i}\lambda(S_2S_{3x} - S_3S_{2x})\sigma_1 - \mathrm{i}\mu(S_3S_{1x} - S_1S_{3x})\sigma_2 \end{aligned} \tag{4}$$

where parameters μ and λ satisfy $\mu^2 = \lambda^2 + 4\rho^2$. If one of them is taken as an independent parameter, the other is a double-valued function of it.

Note the fact that the parameters μ and λ cannot be simultaneously equal to zero; the Lax pair given by (3) and (4) depends on the solutions of equation (1) in any limiting values of the spectral parameter.

In this paper, we introduce an auxiliary parameter k such that

$$\mu = -2\rho\frac{k+k^{-1}}{k-k^{-1}} \qquad \lambda = -2\rho\frac{2}{k-k^{-1}}. \tag{5}$$

Then, in addition to avoiding the necessity of introducing the Riemann surface, the new Lax pair obtained from the old one by an appropriate gauge transformation is indeed independent of particular solutions of the L–L equations in the limit of k tending to ∞, although one does not obtain a new equation.

3. A gauge transformation

The original Lax equations are

$$\partial_x F(k) = L(k)F(k) \qquad \partial_t F(k) = M(k)F(k). \tag{6}$$

We define a gauge transformation [17]

$$F(x,k) = K(x)F'(x,k) \tag{7}$$

where $K(x)$ is independent of k, such that

$$\partial_x F'(x,k) = L'(x,k)F'(x,k) \tag{8}$$

in which $L'(x,k)$ has the property that $L'(x,k) \to 0$, as $k \to \infty$.

One can find

$$K_x(x) = \mathrm{i}2\rho S_2(x)\sigma_2 K(x) \tag{9}$$

and then

$$K(x) = \mathrm{e}^{\mathrm{i}\frac{1}{2}\Omega(x)\sigma_2} \tag{10}$$

where $K(x)$ means a rotation around the 2-axis in the spin space and $\Omega(x)$ is real and denotes a rotation angle around the 2-axis. We also obtain

$$L'(x,k) = -\mathrm{i}(\mu+2\rho)S_2'\sigma_2 - \mathrm{i}\lambda(S_3'\sigma_3 + S_1'\sigma_1) \tag{11}$$

and the prime denotes the rotated quantities. Since (2), the rotation does not affect the asymptotic spin.

We now consider the new Lax equation (8) and neglect the prime. Its asymptotic Jost solution is obviously $\mathrm{e}^{-\mathrm{i}\lambda x\sigma_3}$. The Jost solutions of (8), $(\tilde{\psi}(x,k)\psi(x,k))$ and $(\phi(x,k)\tilde{\phi}(x,k))$, is defined by the asymptotic conditions tending to $\mathrm{e}^{-\mathrm{i}\lambda x\sigma_3}$ as $x \to \pm\infty$. For real k, (8) has two independent solutions with two components; as usual, one has for example

$$\phi(x,k) = a(k)\tilde{\psi}(x,k) + b(k)\psi(x,k) \tag{12}$$

where $a(k)$ and $b(k)$ are independent of x, and

$$a(k) = -\mathrm{i}\phi(x,k)^{\mathrm{T}}\sigma_2\psi(x,k) \qquad b(k) = -\mathrm{i}\tilde{\psi}(x,k)^{\mathrm{T}}\sigma_2\phi(x,k). \tag{13}$$

It is easily seen that $\psi(x,k)$ and $\phi(x,k)$ are able to analytically continue into the region $\mathrm{Im}\lambda > 0$, that is the upper half plane of complex k, by (5). Hence, $a(k)$ can be analytically continued into the upper half plane of complex k. Analogously, $\tilde{\psi}(x,k)$ and $\tilde{\phi}(x,k)$ go analytically into the region $\mathrm{Im}\lambda < 0$, the lower half plane of complex k. In the complex plane, we also have

$$\tilde{\psi}(x,k) = \mathrm{i}\sigma_2\overline{\psi(x,k)} \qquad \tilde{\phi}(x,k) = -\mathrm{i}\sigma_2\overline{\phi(x,k)}. \tag{14}$$

4. The reduction transformation properties

From (5) and (3) one can see that $\overline{L(-\bar{k})} = L(k)$. So we obtain

$$\psi(x,k) = \overline{\psi(x,-\bar{k})} \qquad \phi(x,k) = \overline{\phi(x,-\bar{k})} \qquad a(-\bar{k}) = \overline{a(k)} \tag{15}$$

and

$$b(-k) = \overline{b(k)} \qquad \text{where } k \text{ is real.} \tag{16}$$

The properties of the Jost solutions, $a(k)$ and $b(k)$, under the transformation $k \to -\bar{k}$ are referred to as the reduction transformation properties.

From (15) we can see that if k_n is a zero of $a(k)$, then $-\bar{k}_n$ is also a zero of $a(k)$. Therefore, zeros of $a(k)$ exist in pairs. Sometimes, we write

$$k_{\bar{n}} = -\bar{k}_n \qquad \text{or} \qquad k_{N+n} = -\bar{k}_n. \tag{17}$$

If k_n is a zero of $a(k)$, from (13) we have

$$\phi(x,k_n) = b_n \psi(x,k_n) \tag{18}$$

where b_n is independent of x. Similarly, we have

$$\phi(x,k_{\bar{n}}) = b_{\bar{n}} \psi(x,k_{\bar{n}}) \qquad b_{\bar{n}} = \bar{b}_n. \tag{19}$$

From (15) we have

$$c_n = -\bar{c}_{\bar{n}} \qquad n = 1,2,\ldots,N \tag{20}$$

where

$$c_n = \frac{b_n}{\dot{a}(k_n)} \qquad \dot{a}(k_n) = \frac{\mathrm{d}}{\mathrm{d}k} a(k)_{k=k_n} \qquad n = 1,2,\ldots,2N. \tag{21}$$

The standard procedure then yields

$$a(k) = \prod_{n=1}^{N} \frac{k-k_n}{k-\bar{k}_n} \frac{k+\bar{k}_n}{k+k_n} \exp\left\{ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}k' \frac{\ln(1-|b(k')|^2)}{k'-k} \right\}. \tag{22}$$

5. An inverse scattering transform

Similar to that of the inverse scattering transform for the NLS equation developed in the well known paper of Zakharov and Shabat [19], one can obtain an inverse scattering equation of Zakharov–Shabat form

$$\tilde{\psi}(x,k) = D_{\cdot 1}(x,k)\,\mathrm{e}^{-\mathrm{i}\lambda x} \tag{23}$$

where

$$D_{\cdot 1}(x,k) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} + R_{\cdot 1}(x,k) + J_{\cdot 1}(x,k) \tag{24}$$

$$R_{\cdot 1}(x,k) = \sum_{n=1}^{2N} \frac{1}{k-k_n} \frac{b_n}{\dot{a}(k_n)} \psi(x,k_n)\,\mathrm{e}^{\mathrm{i}\lambda_n x} \tag{25}$$

and

$$J_{\cdot 1}(x,k) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}k' \frac{1}{k'-k} \frac{b(k')}{a(k')} \psi(x,k')\,\mathrm{e}^{\mathrm{i}\lambda' x}. \tag{26}$$

Similarly, we obtain

$$\tilde{\phi}(x,k) = D_{\cdot 2}(x,k)\,\mathrm{e}^{\mathrm{i}\lambda x} \tag{27}$$

where

$$D_{\cdot 2}(x,k) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} + R_{\cdot 2}(x,k) + J_{\cdot 2}(x,k) \tag{28}$$

$$R_{\cdot 2}(x,k) = \sum_{n=1}^{2N} \frac{1}{k-k_n} \frac{1}{\dot{a}(k_n)} \psi(x,k_n)\, \mathrm{e}^{-\mathrm{i}\lambda_n x} \tag{29}$$

and

$$J_{\cdot 2}(x,k) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}k' \frac{1}{k'-k} \frac{\overline{b(k)}}{a(k')} \phi(x,k')\, \mathrm{e}^{-\mathrm{i}\lambda' x}. \tag{30}$$

Although the resulting expression of $M(x,k)$ transformed by the gauge transformation $K(x)$ is more complicated, its asymptotic expression in the limit of $|x| \to \infty$ is simply $\mathrm{i}2\lambda\mu\sigma_3$. By standard procedure, the time dependence is simply achieved by the following replacements:

$$a(k) \to a(t,k) = a(0) \tag{31}$$

$$b(k) \to b(t,k) = b(0,k)\, \mathrm{e}^{\mathrm{i}4\lambda\mu t} \tag{32}$$

and

$$b_n \to b_n(t) = b_n(0)\, \mathrm{e}^{\mathrm{i}4\lambda_n\mu_n t} \tag{33}$$

where $a(0,k)$, etc, are constants.

6. Determination of the gauge transformation

We now determine expression of the gauge transformation. Writing

$$F(x,k) = K(x)(\tilde{\psi}(x,k)\tilde{\phi}(x,k)) \tag{34}$$

and substituting it into (6), we have

$$\partial_x\{K(x)D(x,k)\,\mathrm{e}^{-\mathrm{i}\lambda x\sigma_3}\} = L(x,k)\{K(x)D(x,k)\,\mathrm{e}^{-\mathrm{i}\lambda x\sigma_3}\}. \tag{35}$$

In the limit of $|k| \to \infty$, we obtain

$$\partial_x K(x) = \mathrm{i}2\rho S_2(x)\sigma_2 K(x). \tag{36}$$

In the limit of $|k| \to 0$, we have

$$\partial_x\{K(x)D(x,0)\} = -\mathrm{i}2\rho S_2(x)\sigma_2\{K(x)D(x,0)\}. \tag{37}$$

Comparing these two equations, we find

$$K^{-1}(x) = K(x)D(x,0) \tag{38}$$

or

$$D(x,0) = K^{-2}(x) = K(x)^{\dagger 2} \tag{39}$$

$$K(x)^2 = D(x,0)^{\dagger}. \tag{40}$$

From (10), we have

$$K(x)^2 = \mathrm{e}^{\mathrm{i}\frac{1}{2}2\Omega(x)\sigma_2} = \cos\{\Omega(x)\} + \mathrm{i}\sigma_2 \sin\{\Omega(x)\} \tag{41}$$

whose elements are real and the determinant is unity. This means that

$$D(x,0)_{jk} \text{ is real} \qquad j,k = 1,2 \tag{42}$$

$$D(x,0)_{11} = D(x,0)_{22} \qquad D(x,0)_{21} = -D(x,0)_{12} \tag{43}$$

and

$$\det D(x, 0) = 1. \tag{44}$$

However, only (42) is essential, (43) and (44) are naturally true in the case of (42).

We now determine the original spin, namely, the spin without the gauge transformation. Taking the limit of $k \to 1$, from (35) we obtain

$$(\boldsymbol{S} \cdot \boldsymbol{\sigma}) = G(x, 1)\sigma_3 G(x, 1)^{-1} = G(x, 1)\sigma_3 G(x, 1)^{\dagger} \tag{45}$$

where

$$G(x, k) = K(x)D(x, k). \tag{46}$$

(45) can be explicitly written as

$$\begin{aligned}(S_1)_3 = {} & D_1(0)_{11}\{D_1(1)_{11}\overline{D_1(1)}_{11} - D_1(1)_{21}\overline{D_1(1)}_{21}\} + D_1(0)_{21}\{D_1(1)_{11}\overline{D_1(1)}_{21} \\ & +\overline{D_1(1)}_{11}D_1(1)_{21}\}\end{aligned} \tag{47}$$

$$\begin{aligned}(S_1)_1 = {} & D_1(0)_{11}\{D_1(1)_{11}\overline{D_1(1)}_{21} + \overline{D_1(1)}_{11}D_1(1)_{21}\} - D_1(0)_{21}\{D_1(1)_{11}\overline{D_1(1)}_{11} \\ & -D_1(1)_{21}\overline{D_1(1)}_{21}\}\end{aligned} \tag{48}$$

$$(S_1)_2 = -\mathrm{i}\{D_1(1)_{11}\overline{D_1(1)}_{21} - \overline{D_1(1)}_{11}D_1(1)_{21}\}. \tag{49}$$

We have seen that the second component of $\boldsymbol{S}$ is unaffected by the gauge transformation.

7. A system of linear algebraic equations

In the reflectionless case, setting $k = \bar{k}_m$, from (23) we have

$$\overline{\psi_2(x, k_m)} = \mathrm{e}^{-\mathrm{i}\bar{\lambda}_m x} + \sum_{n=1}^{2N} \frac{1}{\bar{k}_m - k_n} c_n \psi_1(x, k_n)\, \mathrm{e}^{\mathrm{i}(\lambda_n - \bar{\lambda}_m)x} \tag{50}$$

and

$$-\psi_1(x, k_m) = \sum_{n=1}^{2N} \frac{1}{k_m - \bar{k}_n} \bar{c}_n \overline{\psi_2(x, k_n)}\, \mathrm{e}^{-\mathrm{i}(\bar{\lambda}_n - \lambda_m)x}. \tag{51}$$

These equations are similar to those of the MKdV equation relating to N breather solutions, so we can solve it explicitly by a well considered procedure [18]. From (24) we have

$$D(x, 1)_{11} = 1 + \sum_{n=1}^{2N} \frac{1}{1 - k_n} c_n \psi_1(x, k_n)\, \mathrm{e}^{\mathrm{i}\lambda_n x} \tag{52}$$

and

$$D(x, 1)_{21} = \sum_{n=1}^{2N} \frac{1}{1 - k_n} c_n \psi_2(x, k_n)\, \mathrm{e}^{\mathrm{i}\lambda_n x}. \tag{53}$$

Defining

$$h_n = b_n^{\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}\lambda_n x} \tag{54}$$

$$f_n = \dot{a}(k_n)^{-\frac{1}{2}} h_n = c_n^{\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}\lambda_n x} \tag{55}$$

$$\Psi_{jn} = c_n^{\frac{1}{2}} \psi_j(\lambda_n) \tag{56}$$

$$Q_{nm} = \bar{f}_n \frac{1}{\bar{k}_n - k_m} f_m \tag{57}$$

(50) and (51) can be written in matrix form as

$$\Psi_1 = \overline{\Psi}_2 Q \qquad \overline{\Psi}_2 = \bar{f} + \Psi_1 Q^{\mathrm{T}}. \tag{58}$$

Hence, we have

$$\overline{\Psi}_2 = \bar{f}(I - QQ^{\mathrm{T}})^{-1} \qquad \Psi_1 = \bar{f}(I - QQ^{\mathrm{T}})^{-1} Q. \tag{59}$$

For real k_0, we write

$$D(x, k_0)_{11} = 1 - \sum_{n=1}^{2N} \Psi_{1n} g_n(k_0) = 1 - \Psi_1 g(k_0)^{\mathrm{T}} \tag{60}$$

$$\overline{D(x, k_0)}_{21} = -\sum_{n=1}^{2N} \overline{\Psi}_{2n} \bar{g}(k_0)_n = -\overline{\Psi}_2 \bar{g}(k_0)^{\mathrm{T}} \tag{61}$$

where $g_n(k_0) = (1/(k_n - k_0)) f_n$.

With (59), we obtain

$$\overline{D(x, k_0)}_{21} = -\bar{f}(I - QQ^{\mathrm{T}})^{-1} \bar{g}(k_0)^{\mathrm{T}} = \frac{\det(I - QQ^{\mathrm{T}} - \bar{g}(k_0)^{\mathrm{T}} \bar{f})}{\det(I - QQ^{\mathrm{T}})} - 1. \tag{62}$$

$$D(x, k_0)_{11} = 1 - \bar{f}(I - QQ^{\mathrm{T}})^{-1} Q g(k_0)^{\mathrm{T}} = \left(\frac{\det(I - QQ^{\mathrm{T}} - Q g(k_0)^{\mathrm{T}} \bar{f})}{\det(I - QQ^{\mathrm{T}})} \right). \tag{63}$$

Finally, setting $k_0 = 1$, we obtain the expressions for $D(x, 1)$ as needed in (45), etc.

8. Explicit expressions in the case of N

We write

$$D(x, k_0)_{11} = \frac{A(k_0)_N}{C_N} \qquad \overline{D(x, k_0)}_{21} = \frac{B(k_0)_N}{C_N}. \tag{64}$$

These formulae can be calculated by using the known Binet–Cauchy formula (see Appendix for details). The procedure is the same as that leading to the explicit expressions for multi-soliton solutions of the NLS equation and the MNLS equation. We have

$$\begin{aligned} C_N = \det(I - QQ^{\mathrm{T}}) = 1 + \sum_{r=1}^{2N} (-1)^r \\ \times \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant 2N} \sum_{1 \leqslant m_1 < m_2 < \cdots < m_r \leqslant 2N} C(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \end{aligned} \tag{65}$$

$$\begin{aligned} C(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) = \prod_n \prod_m \bar{f}_n^2 f_m^2 (\bar{k}_n - k_m)^{-2} \\ \times \prod_{n<n'} \prod_{m<m'} (\bar{k}_n - \bar{k}_{n'})^2 (k_{m'} - k_m)^2 \end{aligned} \tag{66}$$

where n, n', m, m' satisfy $n, n' \in \{n_1, n_2, \ldots, n_7\}$, $m, m' \in \{m_1, m_2, \ldots, m_7\}$.

Similarly, we write

$$\begin{aligned} B_N(k_0) &= \det(I - QQ^{\mathrm{T}} - \bar{g}(k_0)^{\mathrm{T}} \bar{f}) - \det(I - QQ^{\mathrm{T}}) \\ &= \sum_{r=1}^{2N} (-1)^r \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant 2N} \sum_{1 \leqslant m_2 < \cdots < m_r \leqslant 2N} B(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) \end{aligned} \tag{67}$$

$$\begin{aligned} & B(n_1, n_2 \ldots, n_r; 0, m_2, \ldots, m_r) \\ &= \prod_n \prod_m \frac{k_0 - k_m}{\bar{k}_n - k_0} \bar{f}_n^2 f_m^2 (\bar{k}_n - \bar{k}_m)^{-2} \prod_{n<n'} \prod_{m<m'} (\bar{k}_n - \bar{k}_{n'})^2 (k_{m'} - k_m)^2 \end{aligned} \tag{68}$$

but where n, n', m, m' satisfy $n, n' \in \{n_1, n_2, \ldots, n_r\}$, $m, m' \in \{m_2, \ldots, m_r\}$.

We also write

$$\begin{aligned} A_N(k_0) &= \det(I - QQ^{\mathrm{T}} - Q_g(k_0)^{\mathrm{T}} \bar{f}) \\ &= 1 + \sum_{r=1}^{2N} (-1)^r \\ &\quad \times \sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant 2N} \; \sum_{1\leqslant m_1<m_2<\cdots<m_r\leqslant 2N} A(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \end{aligned} \tag{69}$$

$$\begin{aligned} &A(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \\ &\quad = \prod_n \prod_m \frac{k_0 - \bar{k}_n}{k_0 - k_m} \bar{f}_n^2 f_m^2 (\bar{k}_n - k_m)^{-2} \prod_{n<n'} \prod_{m<m'} (\bar{k}_n - \bar{k}_{n'})^2 (k_{m'} - k_m)^2 \end{aligned} \tag{70}$$

where n, n', m, m' also satisfy $n, n' \in \{n_1, n_2, \ldots, n_r\}$, $m, m' \in \{m_1, m_2, \ldots, m_r\}$.

From (20), (35) and (43), we have

$$C(\check{n}_1, \check{n}_2, \ldots, \check{n}_r; \check{m}_1, \check{m}_2, \ldots, \check{m}_r) = \overline{c(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)} \tag{71}$$

hence C_N is real. Similarly, $B_N(0)$ and $A_N(0)$ are real. (42) is valid.

9. An explicit expression of the 1-soliton solution

When $N = 1$, we obtain expressions of the original spin,

$$(S_1)_3 = 1 - 2\left(\frac{4k_1''^2}{|1-k_1^2|^2} + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1\right)\left(\cosh^2\Theta_1 + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1\right)^{-1} \tag{72}$$

$$\begin{aligned} (S_1)_1 &= \left(2\frac{4k_1''^2}{|1-k_1^2|^2}\sinh\Theta_1\cos\Phi_1 - 2\frac{k_1''}{k_1'}\frac{1-|k_1|^4}{|1-k_1^2|^2}\cosh\Theta_1\sin\Phi_1\right) \\ &\quad \times \left(\cosh^2\Theta_1 + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1\right)^{-1} \end{aligned} \tag{73}$$

$$\begin{aligned} (S_1)_2 &= \left(2\frac{2k_1''(1-|k_1|^2)}{|1-k_1^2|^2}\cosh\Theta_1\cos\Phi_1 + 2\frac{2k_1''^2(1+|k_1|^2)}{k_1'|1-k_1^2|^2}\sinh\Theta_1\sin\Phi_1\right) \\ &\quad \times \left(\cosh^2\Theta_1 + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1\right)^{-1} \end{aligned} \tag{74}$$

where

$$\Phi_1 = 2\lambda_1' x - 2(\lambda_1'\mu_1' - \lambda_1''\mu'' - n)t + \Phi_{10} \tag{75}$$

$$\Theta_1 = 2\lambda_1''(x - V_1 t - x_1) \qquad V_1 = \mu_1' + \frac{\lambda_1'}{\lambda_1''}\mu_1'' \tag{76}$$

and Φ_{10} and x_1 are real constants.

The expressions of the 1-soliton solution for a spin chain with an easy plane (the 31-plane and the asymptotic spin along the 3-axis) are equivalent to those found recently by the method of the Darboux transformation [13] but they have not been found previously by any means.

These expressions depend essentially on two parameters, namely, the two velocities in (75) and (76), which describe a spin configuration, deviating from homogeneous magnetization. The centre of inhomogeneity moves with a constant velocity, while the shape of the soliton (the direction of magnetization in its centre) also changes with another velocity. The expressions cannot be obviously factorized in forms of separated variables, even in moving coordinates. Hence, it is hopeless to solve the L–L equation for a spin chain with an easy plane by means of separation of variables. Moreover, these properties remain even in the approximation of order ρ^2; all attempts to use this approximation failed. It is obvious that, when $\rho \to 0$, these three expressions recover those for the isotropic chain.

Some years ago, in the work of Nakamura and Sasada [7], gauge equivalence of the spin chain with an easy plane to the NLS equation with a repulsive interaction in the case of a non-vanishing boundary value was indicated. As is shown in this work, multi-soliton solutions of the L–L equation for a spin chain with an easy plane tend to those for the isotropic spin chain as anisotropy vanishes. On the other hand, the NLS equation with a repulsive interaction in the case of non-vanishing boundary value has dark soliton solutions; when the boundary value tends to zero, the equation has no non-trivial solutions except zero. Hence, the gauge equivalence between these two equations cannot exist.

Since we have obtained (67) and (69), by using standard procedure, we can find the asymptotic behaviours of the N-soliton solution simply.

10. Concluding remark

For the L–L equation for a continuous spin chain with an easy plane, exact soliton solutions have never been found in the last 20 years by all means tried. An exact single soliton solution with spin non-flip was first given in a previous note [13]. The present work gives a suitable inverse scattering transform, the multi-soliton solutions formally and an explicit expression of the 1-soliton solution. The present work provides a solid basis for further research.

Acknowledgments

The authors would like to thank Professors C H Gu and F K Pu for their valuable suggestions. The project is supported by National Natural Science Foundation of China and the Chinese National Fund for Nonlinear Science Research.

Appendix. Explicit expressions of $\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{k}_0)$

Neglecting k_0 for simplicity, we write

$$\overline{D}_{21} = \frac{\det(I + R')}{\det(I + R)} - 1 \tag{A.1}$$

where $R = -QQ^{\mathrm{T}}$, $R' = R - \bar{g}^{\mathrm{T}}\bar{f}$.

$\det(I + R)$ can be expanded as

$$\det(I + R) = 1 + \sum_{r=1}^{2N} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant 2N} R(n_1, n_2, \ldots, n_r) \tag{A.2}$$

where $R(n_1, n_2, \ldots, n_r)$ is a principal minor, that is a determinant of a submatrix of R, by crossing off the other columns and rows with the $(n_1, n_2, \ldots, n_r)$th columns and $(n_1, n_2, \ldots, n_r)$th rows remaining. By means of the Binet–Cauchy formula, we have

$$R(n_1, n_2, \ldots, n_r) = (-1)^r \sum_{1 \leqslant m_1 < m_2 < \cdots < m_r \leqslant 2N} Q(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)^2. \tag{A.3}$$

On account of the special form of Q_{mn}, (57), we have

$$Q(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) = \prod_n \prod_m \bar{f}_n f_m (\bar{k}_n - k_m)^{-1} \prod_{n<n'} \prod_{m<m'} (\bar{k}_n - \bar{k}_{n'})(k_{m'} - k_m) \tag{A.4}$$

where $n, n' \in \{n_1, n_2, \ldots, n_r\}$, $m, m' \in \{m_1, m_2, \ldots, m_r\}$.

To evaluate $\det(I + R')$, we write $R' = -Q' Q''^{\mathrm{T}}$, where Q' is a $2N \times (2N+1)$ matrix whose elements are

$$Q'_{n0} = \bar{g}_n \qquad Q''_{n0} = \bar{f}_n \qquad n = 1, 2, \ldots, 2N \tag{A.5}$$

$$Q'_{nm} = Q_{nm} \qquad n, m = 1, 2, \ldots, 2N. \tag{A.6}$$

By means of the Binet–Cauchy formula, we have

$$R'(n_1, n_2, \ldots, n_r) = (-1)^r \sum_{0 \leqslant m_1 < m_2 < \cdots < m_r \leqslant 2N} Q'(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \times Q''(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r). \tag{A.7}$$

The summation on the right-hand side is obviously decomposed into two parts: one of $m_1 = 0$ and one of $m_1 \geqslant 1$. The second part is just $R(n_1, n_2, \ldots, n_r)$. Hence, we obtain

$$\det(I + R') - \det(I + R) = \sum_{r=1}^{2N} (-1)^r \times \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant 2N} \sum_{1 \leqslant m_2 < \cdots < m_r \leqslant 2N} Q'(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) \times Q''(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r). \tag{A.8}$$

By means of the similar procedure, we find

$$Q''(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) = \prod_n \prod_m \bar{f}_n f_m (\bar{k}_n - k_m)^{-1} \times \prod_{n<n'} \prod_{m<m'} (\bar{k}_n - \bar{k}_{n'})(k_{m'} - k_m) \tag{A.9}$$

$$Q'(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) = \prod_n \prod_m \frac{(k_0 - k_m)}{\bar{k}_n - k_0} \bar{f}_n f_m (\bar{k}_n - k_m)^{-1} \times \prod_{n<n'} \prod_{m<m'} (\bar{k}_n - \bar{k}_{n'})(k_{m'} - k_m) \tag{A.10}$$

where $n, n' \in \{n_1, n_2, \ldots, n_r\}$, $m, m' \in \{m_2, \ldots, m_r\}$.

Hence, we have obtained the explicit expressions of $\overline{D(x, k_0)_{21}}$.

Equation (63) can be simply written as

$$D_{11} = \frac{\det(I - Q Q'''^{\mathrm{T}})}{\det(I - Q Q^{\mathrm{T}})} \tag{A.11}$$

where

$$Q''' = Q + \bar{f}^{\mathrm{T}} g \qquad \text{or} \qquad Q'''_{nm} = \bar{f}_n \left\{ \frac{1}{\bar{k}_n - k_m} \frac{k_0 - \bar{k}_n}{k_0 - k_m} \right\} f_m. \tag{A.12}$$

A similar procedure yields

$$Q'''(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) = \prod_n \prod_m \frac{k_0 - \bar{k}_n}{k_0 - k_m} \bar{f}_n f_m (\bar{k}_n - k_m)^{-1} \times \prod_{n<n'} \prod_{m<m'} (\bar{k}_n - \bar{k}_{n'})(k_{m'} - k_m). \tag{A.13}$$

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